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MATB44 Week 4 Notes

1. Review of Week 3 Material

a) Homogeneous Eqns With Constant Coefficients

- Rule 1: $y = e^{rt}$
- Rule 2: We can combine 2 solns to get a new soln.

b) Homogeneous Eqns With Constant Coefficients and 2 Real Distinct Roots

- Suppose $ay'' + by' + cy = 0$.

From rule 1, we know that $y = e^{rt}$.

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$ar^2(e^{rt}) + br(e^{rt}) + c(e^{rt}) = 0$

$(e^{rt})(ar^2 + br + c) = 0$

Since $e^{rt} \neq 0$, we can divide both sides by it.

$ar^2 + br + c = 0$ ← Called characteristic equation

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

From rule 2, we know that we can combine 2 solns to get a new one.

$y = c_1 y_1 + c_2 y_2$

c) Wronksian

- Useful for finding the second soln in homogeneous eqns with constant coefficients and repeated roots as well as determining if a pair of solns is a fundamental set of soln.
- Consider $p(t)y'' + q(t)y' + r(t)y = 0$ and $y(t_0) = y_0$ and $y'(t_0) = y'_0$. Suppose y_1 and y_2 are solns. Then, $y = c_1 y_1 + c_2 y_2$ is also a soln. We want to know if $y = c_1 y_1 + c_2 y_2$ is also a general soln. In order to be a general soln, it must satisfy the initial conditions.

$$y_0 = c_1 y_1(t_0) + c_2 y_2(t_0)$$
$$y'_0 = c_1 y'_1(t_0) + c_2 y'_2(t_0)$$

We can use Cramer's rule to find c_1 and c_2.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

The denominator is the Wronksian.
I.e. If y_1 and y_2 are 2 solns to a linear homogeneous eqn with constant coefficients, then

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$
$$= y_1 y'_2 - y'_1 y_2$$

Notice that the only thing preventing us from finding C_1 and C_2 is if $W=0$.
If $W \neq 0$ and y_1 and y_2 are solns, then y_1 and y_2 are a **fundamental set/pair of solns** and $y = C_1 y_1 + C_2 y_2$ is a general soln.

I.e. Take $y = C_1 y_1 + C_2 y_2$ and the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

$$\left.\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= y_0 \\ C_1 y'_1(t_0) + C_2 y'_2(t_0) &= y'_0 \end{aligned}\right\}$$

This system has a soln for any RHS iff $W \neq 0$.

Note: Recall from linear algebra that a matrix is linearly independent iff the determinant of the matrix $\neq 0$. Hence, if $W \neq 0$, y_1 and y_2 are linearly independent. otherwise, they are linearly dependent.

Note: If $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$, then y_1 and y_2 are a fundamental pair of solns.

E.g. 1 Let $f(t) = e^{2t}$, $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 3e^{4t}$.

Solve for $g(t)$.

Soln:

$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

$fg' - f'g = 3e^{4t}$

$e^{2t} g' - 2e^{2t} g = 3e^{4t}$

$e^{2t}(g'-2g)=3e^{4t}$

$g'-2g=3e^{2t}$ ← Linear first order

$g=3te^{2t}+\underline{c}e^{2t}$

We keep the c.

- We can solve W without solving the eqn.

$W = y_1y_2' - y_1'y_2$

$$\frac{dW}{dt} = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2$$

$$= y_1y_2'' - y_1''y_2$$

Recall that

$y_1'' + p(t)y_1' + q(t)y_1 = 0 \Rightarrow y_1'' = -p(t)y_1' - q(t)y_1$

$y_2'' + p(t)y_2' + q(t)y_2 = 0 \Rightarrow y_2'' = -p(t)y_2' - q(t)y_2$

$$\frac{dW}{dt} = y_1(-p(t)y_2' - q(t)y_2) - y_2(-p(t)y_1' - q(t)y_1)$$

$$= -p(t)y_1y_2' - q(t)y_1y_2 + p(t)y_1'y_2 + q(t)y_1y_2$$

$$= p(t)(-y_1y_2' + y_1'y_2)$$

$$= -p(t)(y_1y_2' - y_1'y_2)$$

$$= -p(t)W$$

$$\frac{1}{W}\,dW = -p(t)\,dt$$

$$\int \frac{1}{W}\,dW = \int -p(t)\,dt$$

$$\ln(W) + c = \int -p(t)\,dt$$

$$\ln(W) = \int -p(t)\,dt + c$$

$$W = e^{-\int p(t)dt + c}$$

$$= e^{-\int p(t)dt} \cdot e^{c}$$

$$= c'e^{-\int p(t)dt}$$ ← Abel's Formula

– The **Wronksian Dichotomy for 2 Solns** states that for 2 solns, $W=0$ for all t or $W \neq 0$ for all t.

Abel's formula proves the dichotomy. Either $c'=0$ and $W=0$ everywhere or $c' \neq 0$ and $W \neq 0$ everywhere.

E.g. 2 Let $x^2y'' - x(x+2)y' + (x+2)y = 0$

a) Verify $y_1 = x$ and $y_2 = xe^x$

Soln:

Simply plug y_1 and y_2 into the eqn above and see if LHS = RHS.

y_1:

$$\begin{aligned} LHS &= -x(x+2) + (x+2)x \\ &= 0 \end{aligned}$$

$RHS = 0$

$LHS = RHS$

y_2:

$$\begin{aligned} LHS &= x^2(xe^x)'' - x(x+2)(xe^x)' + (x+2)(xe^x) \\ &= x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)(x)(e^x) \\ &= e^x\left(x^3 + 2x^2 - x^3 - 3x^2 - 2x + x^2 + 2x\right) \\ &= e^x(0) \\ &= 0 \end{aligned}$$

$RHS = 0$

$LHS = RHS$

b) Do y_1 and y_2 make a fundamental pair of solns?

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix}$$

$$= x(e^x + xe^x) - xe^x$$

$$= xe^x + x^2e^x - xe^x$$

$$= x^2e^x$$

$W \neq 0$ Iff $x \neq 0$

Note: We can't use Abel's formula here. All it says is that $W=0$ iff $c'=0$, which doesn't give us enough information.

E.g. 3 Suppose we have $y_1 = e^{-t}$ and $y_2 = 2e^{-t}$. Prove that they are not a fundamental pair of solns.

Solns:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-t} & 2e^{-t} \\ -e^{-t} & -2e^{-t} \end{vmatrix}$$

$$= (e^{-t})(-2e^{-t}) - (2e^{-t})(-e^{-t})$$

$$= 0$$

E.g. 4 Let $y_1 = t$ and $y_2 = \sin t$. Are y_1 and y_2 a fundamental pair of solns?

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} = t\cos t - \sin t$$

If $t=0$, $W=0$.

Furthermore, $\left.\begin{aligned} W(\tfrac{\pi}{2}) &= -1 < 0 \\ W(2\pi) &= 2\pi > 0 \end{aligned}\right\}$ Contradiction

Abel's formula either gives all positives or all negatives.
Hence, y_1 and y_2 cannot be a fundamental pair of solns.

d) Homogeneous Eqns with Constant Coefficients and Repeated Roots

- Here $r_1 = r_2$. However, this poses a problem. We need y_1 and y_2, and right now, we just have y_1.
- E.g. 5 Solve $y'' + 2y' + y = 0$

  Soln:

  $r^2 + 2r + 1 = 0$

  $(r+1)^2 = 0$

  $r = -1$

  $y_1 = e^{-t}$

To find y_2, we'll use the Wronksian.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_1' y_2$$

$$= (e^{-t}) y_2' - (-e^{-t}) y_2$$

$$= (e^{-t}) y_2' + (e^{-t}) y_2$$

$$= (e^{-t})(y_2' + y_2)$$

Also, $W = c' \cdot e^{-\int p \, dt}$

$$= c' \cdot e^{-\int 2 \, dt}$$

$$= c' \cdot e^{-2t + c_1}$$

$$= e^{-2t}$$

$(e^{-t})(y_2' + y_2) = e^{-2t}$

$y_2' + y_2 = e^{-t}$ ← Linear Differential eqn

$y_2 = te^{-t}$

Note: If we have $y'' + by' + cy = 0$ and $r_1 = r_2$ then $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$.
This is called the Repeated Roots Rule.

e) Homogeneous Eqns With Constant Coefficients and Complex Roots

- Z is a complex number if it can be written in the form: $z = a + ib$.
  a and b are real numbers.
  $i = \sqrt{-1} \Leftrightarrow i^2 = -1$
  a is the real part.
  b is the imaginary part. (Does not include i).
  i is the imaginary unit.

- $(a+ib)+(c+id)=(a+c)+i(b+d)$

  E.g. 6 $(1+2i)+(3+4i)=4+6i$

- $(a+ib)\times(c+id) = ac + iad + ibc + i^2bd$

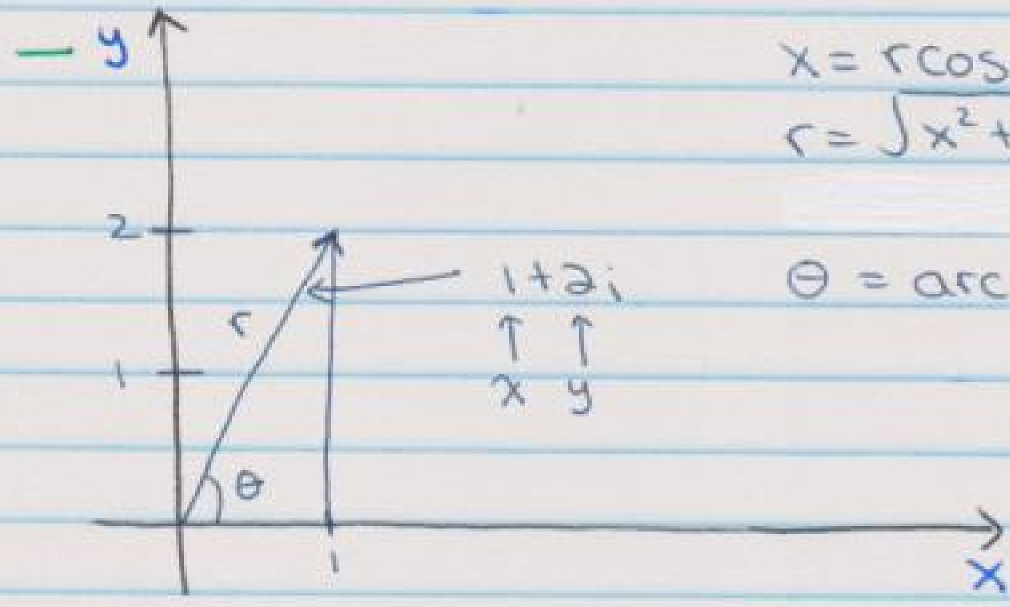
  $= ac + i(ad+bc) - bd$

  $= (ac-bd) + i(ad+bc)$

E.g. 7 $(1+2i)\times(3+4i)$

$= 3+4i+6i+8i^2$

$= -5+10i$

- 

$x = r\cos\theta$, $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

y is the imaginary part.

x is the real part.

$Z = x+iy$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$ ← Polar form of Complex Numbers

- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

  Consider

  $e^{i\alpha} = \cos\alpha + i\sin\alpha$

  $e^{i\beta} = \cos\beta + i\sin\beta$

$$
\begin{aligned}
e^{i\alpha} \times e^{i\beta} &= (\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta) \\
&= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + \\
&\quad i^2 \sin\alpha\sin\beta \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\
&= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\
&= e^{i(\alpha+\beta)}
\end{aligned}
$$

Hence, $e^{a+ib} = e^{a} \cdot e^{ib}$

If $R_1 \neq R_2$ and $R_1, R_2 \in \mathbb{C}$, $R_1, R_2 = \lambda \pm iu$,
$y_1 = e^{\lambda t}\cos(ut)$, $y_2 = e^{\lambda t}\sin(ut)$.

E.g. 8 Solve $y'' + y' + 9.25y = 0$

Soln:

$r^2 + r + 9.25 = 0$

$r = \frac{-1}{2} \pm 3i$

Recall that $y = e^{rt}$

$$
\begin{aligned}
y &= e^{rt} \\
&= e^{(-\frac{1}{2} + 3i)t} \\
&= e^{-\frac{t}{2} + (3t)i} \\
&= e^{-t/2} \cdot e^{(3t)i} \\
&= e^{-t/2}(\cos(3t) + i\sin(3t)) \quad \text{By Euler's Formula} \\
&= \underbrace{e^{-t/2}(\cos(3t))}_{\text{Solns}}, + i\underbrace{e^{-t/2}(\sin(3t))}_{\text{Solns}},
\end{aligned}
$$

Note: We're only taking the real parts.

$e^{-\frac{t}{2}}\cos(3t) = e^{\lambda}\cos(ut)$
$e^{-t/2}\sin(3t) = e^{\lambda}\sin(ut)$

Note: We don't need to consider $r = \frac{-1}{2} - 3i$ because it gives us redundant solns.

$r_2 = \frac{-1}{2} - 3i$

$y = e^{-\frac{t}{2}}((\cos(-3t)) + (i\sin(-3t)))$
$= e^{-\frac{t}{2}}(\cos(-3t)) + ie^{-\frac{t}{2}}(\sin(-3t))$

However, because cos is an even function, $\cos(-3t) = \cos(3t)$.

Furthermore, because sin is an odd function, $\sin(-3t) = -\sin(3t)$.

Hence, we get no new solns.

2 Reduction of Order

- Now we will focus on homogeneous eqns with non-constant coefficients.
- Rule 1: $y_1(t) = \frac{1}{t}$
- Rule 2: $y_2(t) = v(t)\,y_1(t)$ where $v(t)$ is an unknown function.
- E.g. 9 Solve $2t^2y'' + 3ty' - y = 0$

Soln:

$y_1(t) = \frac{1}{t}$

$y_2(t) = v(t)\,y_1(t)$

$y_2' = (vy_1)'$

$= v'y_1 + vy_1'$

$y_2'' = (v'y_1 + vy_1')'$

$= v''y_1 + v'y_1' + v'y_1' + vy_1''$

$= v''y_1 + 2v'y_1' + vy_1''$

Now, put y_2, y_2' and y_2'' back into the original eqn.

$2t^2(v''y_1 + 2v'y_1' + vy_1'') + 3t(v'y_1 + vy_1')$
$- vy_1 = 0$

Now, expand the above eqn.

$2t^2v''y_1 + 4t^2v'y_1' + 2t^2vy_1'' + 3tv'y_1 + 3tvy_1'$
$- vy_1 = 0$

Now, collect all the terms that has v in it.
We do not collect the terms with v' or v''.

$$\underbrace{(2t^2 v y_1'' + 3tvy_1' - vy_1)} + (2t^2 v'' y_1 + 4t^2 v' y_1' + 3tv' y_1) = 0$$

All the terms with a v.

$$v\underbrace{(2t^2 y_1'' + 3ty_1' - y_1)} + (2t^2 v'' y_1 + 4t^2 v' y_1' + 3tv' y_1) = 0$$

Notice that this is our original eqn but with y_1 instead of y. Since we know that y_1 is a soln, then this equals 0.

Note: If you did your calculations correctly, at some point all the terms with v will go away. V must go.

Now, we're left with

$$2t^2 v'' y_1 + 4t^2 v' y_1' + 3tv' y_1 = 0 \Big\} \text{ No term with v.}$$

Now, let $w = v'$ and $w' = v''$.

So, we have: $2t^2 w' y_1 + 4t^2 w y_1' + 3tw y_1 = 0$

Plug in $y_1 = \frac{1}{t} \rightarrow$ $\dfrac{2t^2 w'}{t} + \dfrac{4t^2 w}{-t^2} + \dfrac{3tw}{t} = 0$

$$2tw' - 4w + 3w = 0$$

$$2tw' - w = 0$$

$$2t \frac{dw}{dt} = w$$

$$2t\, dw = w\, dt$$

$$\frac{1}{w}\, dw = \frac{1}{2t}\, dt \leftarrow \text{Separable Eqns}$$

$$\int \frac{1}{w}\, dw = \int \frac{1}{2t}\, dt$$

$$\ln|w| + c_1 = \frac{\ln|t|}{2} + c_2$$

$$\ln|w| = \frac{\ln|t|}{2} + \underbrace{c_2 - c_1}_{c}$$

$$= \frac{\ln|t|}{2} + c$$

$$e^{\ln|w|} = e^{\frac{\ln|t|}{2} + c}$$

$$w = e^{c} \cdot e^{\frac{\ln|t|}{2}}$$

$$= c' \cdot (e^{\ln|t|})^{1/2}$$

$$= c' \cdot t^{1/2}$$

Note: If we just want 1 y_2, we can let $c' = 1$.

$$w = t^{1/2}$$

Now, we will solve for v.

$$v' = w$$

$$v = \int w\, dt$$

$$= \int t^{1/2}\, dt \longrightarrow \int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$$= \frac{2}{3} t^{3/2}$$

$$y_2 = v y_1$$

$$= \left(\frac{2}{3} t^{3/2}\right)\left(\frac{1}{t}\right)$$

$$= \frac{2}{3} t^{1/2}$$

Note: Letting $y_2 = v y_1$ is called D'Alembert's step. When you do it, y_1 and y_2 always make a fundamental pair of solns.

Note: If the question also asks to verify that y_1 and y_2 are a fundamental pair of solns, show that $W \neq 0$.

I.e. In our example $y_1 = \frac{1}{t}$, $y_2 = \frac{2}{3} t^{1/2}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{t} & \frac{2}{3} t^{1/2} \\ -\frac{1}{t^2} & \frac{1}{3} t^{-1/2} \end{vmatrix}$$

$$= \left(\frac{1}{t}\right)\left(\frac{1}{3} t^{-1/2}\right) - \left(-\frac{1}{t^2}\right)\left(\frac{2}{3} t^{1/2}\right)$$

$$= \frac{1}{3t^{3/2}} + \frac{2}{3t^{3/2}}$$

$$= \frac{1}{t^{3/2}}$$

$$\neq 0$$

Note: The steps I did to find y_2 were used that v has to go. On assignments/quizzes/tests, you don't have to show all those steps. Here are the steps you should do on tests/quizzes/assignments.

$$2t^2 \left(v''\left(\tfrac{1}{t}\right) + 2v'\left(\tfrac{1}{t}\right)' + v\left(\tfrac{1}{t}\right)''\right) + 3t\left(v'\left(\tfrac{1}{t}\right) + v\left(\tfrac{1}{t}\right)'\right) -$$
$$v\left(\tfrac{1}{t}\right) = 0$$

Ignore all terms with a v.

Note: If you expand and simplify, you'll find that the terms with v cancels out.

$$\frac{2t^2v''}{t} + \frac{4t^2v'}{-t^2} + \frac{3tv'}{t} = 0$$

$$2tv'' - 4v' + 3v' = 0$$

$$2tv'' - v' = 0$$

Let $w = v'$ and $w' = v''$.

$$2t\frac{dw}{dt} = w$$

$$\frac{1}{w}\,dw = \frac{1}{2t}\,dt$$

$$\int \frac{1}{w}\,dw = \int \frac{1}{2t}\,dt$$

$$\ln|w| + C_1 = \frac{\ln|t|}{2} + C_2$$

$$\ln|w| = \frac{\ln|t|}{2} + C$$

$$w = C' \cdot t^{1/2}$$
$$= t^{1/2}$$

$$v' = w$$
$$v = \int w\,dt$$
$$= \int t^{1/2}\,dt$$
$$= \frac{2}{3}t^{3/2}$$

$$y_2 = \frac{2t^{3/2}}{3} \cdot y_1$$
$$= \frac{2}{3}t^{3/2} \cdot \frac{1}{t}$$
$$= \frac{2}{3}t^{1/2}$$

Note: Another way to solve for y_2 is to use Abel's formula. Here, it is crucial that the coefficient of y'' is 1.

$$2t^2y'' + 3ty' - y = 0$$
$$y'' + \frac{3y'}{2t} - \frac{y}{2t^2} = 0$$

Now, apply Abel's formula.

$$W_1 = C'e^{-\int p(t)\,dt}$$
$$= C'e^{-\int \frac{3}{2t}\,dt}$$
$$= C'e^{-\frac{3}{2}\int \frac{1}{t}\,dt}$$
$$= C'e^{-\frac{3}{2}\ln|t| + C_1}$$
$$= C' \cdot C_1' \cdot t^{-3/2}$$
$$= \frac{C}{t^{3/2}} \leftarrow \text{Take } C=1$$
$$= \frac{1}{t^{3/2}}$$

Another way to compute W.

$$W_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= y_1y_2' - y_1'y_2$$
$$= \frac{y_2'}{t} + \frac{y_2}{t^2}$$

$$W_1 = W_2$$

$$\frac{1}{t^{3/2}} = \frac{y'_2}{t} + \frac{y_2}{t^2}$$

$$y'_2 + \frac{y_2}{t} = \frac{1}{t^{1/2}}$$ ← Linear Diff Eqn First order

$$\mu(y'_2) + \frac{\mu y_2}{t} = \frac{\mu}{t^{1/2}}$$

$$LHS = (\mu y_2)'$$

$$\cancel{\mu(y'_2)} + \frac{\mu y_2}{t} = (\mu y_2)'$$

$$= \mu' y_2 + \cancel{\mu y'_2}$$

$$\frac{\mu \cancel{y_2}}{t} = \mu' \cancel{y_2}$$

$$\frac{1}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{1}{t}\,dt = \ln(\mu)$$

$$\ln|t| + C = \ln(\mu)$$

$$\mu = t$$

$$(\mu y_2)' = \frac{\mu}{t^{1/2}}$$

$$(t y_2)' = \frac{t}{t^{1/2}}$$

$$= t^{1/2}$$

$$t y_2 = \int t^{1/2}\,dt$$

$$= \frac{2}{3} t^{3/2} + C$$

$$y_2 = \frac{2}{3} t^{1/2} + \underbrace{\frac{C}{t}}_{\text{Let this be 0.}} \longrightarrow y_2 = \frac{2}{3} t^{1/2}$$